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# On Riemann's Period matrix of $Y^2 = X^{2n+1} - 1$

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## Abstract

We are deeply interested in the theory of Abelian integrals which have an ample data concerning the moduli space of the Riemann surfaces. The theta-function is a clue to connect the defining equation to the moduli space and the properties of the theta-function are well-known, but the concrete examples of such functions are not known so much.

On the Riemann's period matrix of the Riemann surface  $y^2 = x^{2n+1} - 1$ , the determinant of the above matrix of the general type  $n$  (see [7]) and the above matrix of the type  $n = 3$  (see [8]) are done.

In this paper, we have the Riemann's period matrix of the Riemann surface  $y^2 = x^{2n+1} - 1$ .

## 1 Introduction

In §2, we have the well-known general theorems of Riemann's period matrix of hyper-elliptic functions.

In §3, we show the ways of getting the Riemann's period matrix of hyper-elliptic functions.

The first is to choose a base of the vector spaces of holomorphic 1-form on the Riemann surface  $X$  for  $y^2 = x^{2n+1} - 1$ . The second is to make a model of  $y^2 = x^{2n+1} - 1$ . As a

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topological space,  $X$  is a compact orientable 2-manifold, its genus is  $n$  by Riemann-Roch theorem and its branch points are  $(2n+1)^{th}$  root of 1 and  $\infty$ . The third is to analyze the periods of homology ( $2n$  disjoint simple closed paths with all beginning and ending at the same base point).

Since the periods have to be decided so that Riemann's period relations hold, we determine five rules to get the correct periods from a polygon with  $4n$  sides (one side each for the left and right sides of each path).

In §4, by using the above five rules and Cauchy integral formula, we get the Period matrix of the Riemann surface  $y^2 = x^{2n+1} - 1$  by the values of the branch points.

## 2 General Theory of Hyper-Elliptic Function

### 2.1 Period Matrix and Quasi-Period Matrix

Let  $C$  be the following hyper-elliptic curve

$$C : y^2 = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{2n+1} x^{2n+1} \quad \lambda_i \in \mathbb{C} \quad (i = 0, 1, \dots, 2n+1)$$

Since the genus of  $C$  is  $n$ , the vector space of holomorphic 1-forms is an  $n$ -vector space by Riemann-Roch theorem. In fact the following is such a base.

$$\omega_1 = \frac{1}{y} dx, \quad \omega_2 = \frac{x}{y} dx, \quad \omega_3 = \frac{x^2}{y} dx, \quad \dots, \quad \omega_n = \frac{x^{n-1}}{y} dx$$

And also, homology group on the Riemann's surface for the hyperelliptic curve  $C$  has  $2n$  generator  $A_i, B_i$  ( $i = 1, \dots, n$ ) which satisfy the following conditions.

- 1)  $A_i \times A_j = B_i \times B_j = 0 \quad (i \neq j)$
- 2)  $A_i \times B_j = \delta_{ij} \quad (i, j = 1, \dots, n)$

where  $\times$  means intersection number. So we have a Riemann's period matrix  $\Omega = (\Pi, \Pi')$

from the above.

**Definition 2.1 (Riemann's period matrix)** For a base of holomorphic 1-forms  $\omega_i$  ( $i = 1, \dots, n$ ) and a base of homology  $A_i, B_i$  ( $i = 1, \dots, n$ ) on  $C$ . Riemann's period matrix  $\Omega$  is given as follows,

$$\text{Period matrix: } \Omega = (\Pi, \Pi')$$

$$\Pi = \begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix} \quad \Pi' = \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \cdots & \pi'_{nn} \end{pmatrix}$$

$$\pi_{ij} = \int_{A_j} \omega_i \quad \pi'_{ij} = \int_{B_i} \omega_j$$

**Lemma 2.1 (Riemann's period relation 1)** *Let  $X$  be a compact Riemann surface of genus  $n$ , with canonical dissection  $X = X_0 \cup A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n$ .*

*For any holomorphic 1-form  $\omega_i, \omega_j$  ( $i, j = 1, \dots, n$ ), the periods satisfy the following equation.*

$$\sum_{k=1}^n \left[ \int_{A_k} \omega_i \int_{B_k} \omega_j - \int_{B_k} \omega_i \int_{A_k} \omega_j \right] = 0$$

Proof. Since  $X_0$  is simply connected, there is a holomorphic function  $f$  on  $X_0$  such that  $\omega_i = df$  namely,  $f(z) = \int_{z_0}^z \omega_i$  then  $f\omega_i$  is a closed 1-form, so by Green's theorem

$$\begin{aligned} 0 &= \int_{X_0} d(f\omega_j) \\ &= \int_{\partial X_0} f\omega_j \\ &= \sum_{k=1}^n \left[ -\int_{A_k^+} f\omega_j + \int_{A_k^-} f\omega_j - \int_{B_k^+} f\omega_j + \int_{B_k^-} f\omega_j \right] \\ &= \sum_{k=1}^n \int_{A_k} \left[ (f \text{ on } A_k^-) - (f \text{ on } A_k^+) \right] \omega_j + \sum_{k=1}^n \int_{B_k} \left[ (f \text{ on } B_k^-) - (f \text{ on } B_k^+) \right] \omega_j \end{aligned}$$

As  $df$  has no discontinuity on  $A_k$  or  $B_k$ ,  $f$  on  $A_k^+$  must differ from  $f$  on  $A_k^-$  by constant, and likewise for  $B_k^+, B_k^-$ . But the path  $A_k$  lead from  $A_k^-$  to  $A_k^+$  and the path  $A_k$  lead from  $B_k^+$  to  $B_k^-$ .

Thus

$$\begin{aligned} 0 &= \sum_{k=1}^n \int_{A_k} [f(z^-) - f(z^+)] \omega_j + \sum_{k=1}^n \int_{B_k} [f(\zeta^-) - f(\zeta^+)] \omega_j \\ &= \sum_{k=1}^n \int_{A_k} \left( -\int_{z^-}^{z^+} df \right) \omega_j + \sum_{k=1}^n \int_{B_k} \left( -\int_{\zeta^-}^{\zeta^+} df \right) \omega_j \\ &= \sum_{k=1}^n \int_{A_k} \left( -\int_{z^-}^{t_2} \omega_i - \int_{t_4}^{t_3} \omega_i - \int_{t_3}^{z^+} \omega_i \right) \omega_j \\ &\quad + \sum_{k=1}^n \int_{B_i} \left( -\int_{\zeta^-}^{t_3} \omega_i - \int_{t_3}^{t_2} \omega_i - \int_{t_2}^{\zeta^+} \omega_i \right) \omega_j \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^n \left[ \int_{A_k} (-\int_{B_k} \omega_i) \omega_j \right] + \sum_{k=1}^n \left[ \int_{B_k} (+\int_{A_k} \omega_i) \omega_j \right] \\
&= \sum_{k=1}^n \left[ -(\int_{B_k} \omega_i)(\int_{A_k} \omega_j) \right] + \sum_{i=k}^n \left[ (\int_{A_k} \omega_i)(\int_{B_k} \omega_j) \right]
\end{aligned}$$

which proves the lemma.

**Lemma 2.2 (Riemann's period relation 2)** *Let  $X$  be a compact Riemann surface of genus  $n$ , with canonical dissection  $X = X_0 \cup A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n$ . For any holomorphic 1-forms  $\omega_i$  ( $i = 1, \dots, n$ ), the period satisfy the following equation*

$$i \sum_{k=1}^n \left[ \int_{A_k} \omega_i \int_{B_k} \bar{\omega}_i - \int_{B_k} \omega_i \int_{A_k} \bar{\omega}_i \right] > 0$$

namely,

$$\text{Im} \sum_{k=1}^n \left( \int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i \right) > 0$$

Proof. Likewise proof of lemma 2.1.

$$\begin{aligned}
-i \int_{X_0} d(\bar{f} \omega_i) &= -i \int_{\partial X_0} \bar{f} \omega_i \\
&= -\sum_{k=1}^n \left[ -\int_{A_k^+} \bar{f} \omega_i + \int_{A_k^-} \bar{f} \omega_i - \int_{B_k^+} \bar{f} \omega_i + \int_{B_k^-} \bar{f} \omega_i \right] \\
&= -i \sum_{k=1}^n \left[ \int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i - \int_{B_k} \bar{\omega}_i \int_{A_k} \omega_i \right]
\end{aligned}$$

On the other hand,  $d(\bar{f} \omega) = d\bar{f} \wedge df$ . Wherever  $f$  is a local analytic coordinates, let  $f = x + iy$  and  $x, y$  are real coordinates, then

$$\begin{aligned}
d\bar{f} \wedge df &= (dx - idy) \wedge (dx + idy) = 2idx \wedge dy \\
-i \sum_{k=1}^n \left[ \int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i - \int_{B_k} \bar{\omega}_i \int_{A_k} \omega_i \right] &= 2 \int_{X_0} dx \wedge dy > 0
\end{aligned}$$

So we have

$$\text{Im} \sum_{k=1}^n \left( \int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i \right) > 0$$

which proves lemma 2.2.

q.e.d.

**Theorem 2.1 (Riemann)** For the Riemann's period matrix  $\Omega = (\Pi, \Pi')$  of  $C$

Modular matrix  $T = \Pi^{-1}\Pi'$  is symmetric matrix

Proof. By lemma 2.1, for all  $i, j$  ( $i, j = 1, \dots, n$ ), we have

$$\sum_{k=1}^n [\pi_{ik}\pi'_{jk} - \pi'_{ik}\pi_{jk}] = 0$$

$$\begin{pmatrix} \pi_{i1} & \cdots & \pi_{in} \end{pmatrix} \begin{pmatrix} \pi'_{j1} \\ \vdots \\ \pi'_{jn} \end{pmatrix} - \begin{pmatrix} \pi'_{i1} & \cdots & \pi'_{in} \end{pmatrix} \begin{pmatrix} \pi_{j1} \\ \vdots \\ \pi_{jn} \end{pmatrix} = 0$$

$$\begin{pmatrix} \pi_{i1} & \cdots & \pi_{in} \end{pmatrix}^t \begin{pmatrix} \pi'_{j1} & \cdots & \pi'_{jn} \end{pmatrix} - \begin{pmatrix} \pi'_{i1} & \cdots & \pi'_{in} \end{pmatrix}^t \begin{pmatrix} \pi_{j1} & \cdots & \pi_{jn} \end{pmatrix} = 0$$

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix}^t \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \cdots & \pi'_{nn} \end{pmatrix} - \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \cdots & \pi'_{nn} \end{pmatrix}^t \begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix} = 0$$

$$\Pi^t \Pi' = \Pi'^t \Pi \quad (*)$$

Here, let  $^*\omega_1, \dots, ^*\omega_n$  be another base of holomorphic 1-forms which differ from  $\omega_1, \dots, \omega_n$  such that

$$\begin{pmatrix} ^*\omega_1 \\ \vdots \\ ^*\omega_n \end{pmatrix} = \Lambda \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$$

then the period matrix  $^*\Omega = (^*\Pi, ^*\Pi')$  for a base  $^*\omega_1, \dots, ^*\omega_n$  is the following:

$$^*\pi_{ij} = \int_{A_j} ^*\omega_i = \int_{A_j} \sum_{l=1}^n \lambda_{il} \omega_l = \sum_{l=1}^n \lambda_{il} \int_{A_j} \omega_l = \sum_{l=1}^n \lambda_{il} \pi_{lj}$$

$$^*\pi'_{ij} = \int_{B_j} ^*\omega_i = \int_{B_j} \sum_{l=1}^n \lambda_{il} \omega_l = \sum_{l=1}^n \lambda_{il} \int_{B_j} \omega_l = \sum_{l=1}^n \lambda_{il} \pi'_{lj}$$

$$^*\pi_{ij} = \begin{pmatrix} \lambda_{i1} & \cdots & \lambda_{in} \end{pmatrix} \begin{pmatrix} \pi_{1j} \\ \vdots \\ \pi_{nj} \end{pmatrix} \quad ^*\pi'_{ij} = \begin{pmatrix} \lambda_{i1} & \cdots & \lambda_{in} \end{pmatrix} \begin{pmatrix} \pi'_{1j} \\ \vdots \\ \pi'_{nj} \end{pmatrix}$$

$$\begin{pmatrix} {}^*\pi_{11} & \cdots & {}^*\pi_{1n} \\ \vdots & & \vdots \\ {}^*\pi_{n1} & \cdots & {}^*\pi_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix} : {}^*\Pi = \Lambda\Pi$$

$$\begin{pmatrix} {}^*\pi'_{11} & \cdots & {}^*\pi'_{1n} \\ \vdots & & \vdots \\ {}^*\pi'_{n1} & \cdots & {}^*\pi'_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \cdots & \pi'_{nn} \end{pmatrix} ; {}^*\Pi' = \Lambda\Pi'$$

$${}^*\Omega = ({}^*\Pi, {}^*\Pi') = (\Lambda\Pi, \Lambda\Pi') = \Lambda(\Pi, \Pi') = \Lambda\Omega$$

If  $\Lambda = \Pi^{-1}$  ( $\det \Pi \neq 0$ ), then

$${}^*\Omega = ({}^*\Pi, {}^*\Pi') = (I_n, \Pi^{-1}\Pi') = (I_n, T)$$

Using equation (\*) in lemma 2.2 for  ${}^*\Omega$

$${}^*\Pi^t({}^*\Pi') = {}^*\Pi'^t({}^*\Pi)$$

By  ${}^*\Pi = I_n$  (unit matrix) and  ${}^*\Pi' = T$

$${}^tT = T$$

Theorem 2.1 means that there is a suitable base of holomorphic 1-forms so that  $A$ -period matrix  $\Pi$  is a unit matrix  $I_n$  and  $B$ -period matrix  $\Pi'$  is a symmetric matrix  $T$ .

Thus,  $T = \Pi^{-1}\Pi'$  is a symmetric matrix.

q.e.d.

**Theorem 2.2 (Riemann)** *For the Riemann's period matrix  $\Omega = (\Pi, \Pi')$  of hyper-elliptic curve  $C$*

$$\text{Im}T = \text{Im}(\Pi^{-1}\Pi') \text{ is a real symmetric matrix of positive definite}$$

Proof.

$$-i\bar{\Omega}J^t\Omega = -i \begin{pmatrix} \bar{\Pi} & \bar{\Pi}' \end{pmatrix} \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} \begin{pmatrix} {}^t\Pi \\ {}^t\Pi' \end{pmatrix} = i(\bar{\Pi}'^t\Pi - \bar{\Pi}^t\Pi')$$

From the above equation,  $-i\bar{\Omega}J^t\Omega$  is Hermite matrix. Making a following Hermite form for this matrix,

$$\begin{aligned}
\bar{\lambda}(-i\bar{\Omega}J^t\Omega)^t\bar{\lambda} &= i(\bar{\lambda}_1 \cdots \bar{\lambda}_n)(\bar{\Pi}^t\Pi - \bar{\Pi}^t\Pi')^t(\lambda_1 \cdots \lambda_n) \\
&= i \begin{pmatrix} \bar{\lambda}_1 & \cdots & \bar{\lambda}_n \end{pmatrix} \begin{pmatrix} \bar{\pi}'_{11} & \cdots & \bar{\pi}'_{1n} \\ \vdots & & \vdots \\ \bar{\pi}'_{ni} & \cdots & \bar{\pi}'_{nn} \end{pmatrix} \begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{ni} & \cdots & \pi_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \\
&\quad - i \begin{pmatrix} \bar{\lambda}_1 & \cdots & \bar{\lambda}_n \end{pmatrix} \begin{pmatrix} \bar{\pi}_{11} & \cdots & \bar{\pi}_{1n} \\ \vdots & & \vdots \\ \bar{\pi}_{ni} & \cdots & \bar{\pi}_{nn} \end{pmatrix} \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{ni} & \cdots & \pi'_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \\
&= i \sum_{k=1}^n \left[ \left( \sum_{i=1}^n \lambda_i \pi_{ik} \right) \left( \sum_{j=1}^n \bar{\lambda}_j \pi'_{jk} \right) - \left( \sum_{i=1}^n \lambda_i \pi'_{ik} \right) \left( \sum_{j=1}^n \bar{\lambda}_j \pi_{jk} \right) \right] \\
&= i \sum_{k=1}^n \left[ \left( \sum_{i=1}^n \lambda_i \int_{A_k} \omega_i \right) \left( \sum_{j=1}^n \bar{\lambda}_j \int_{B_k} \bar{\omega}_j \right) - \left( \sum_{i=1}^n \lambda_i \int_{B_k} \omega_i \right) \left( \sum_{j=1}^n \bar{\lambda}_j \int_{A_k} \bar{\omega}_j \right) \right] \\
&= i \sum_{k=1}^n \left[ \int_{A_k} \left( \sum_{i=1}^n \lambda_i \omega_i \right) \int_{B_k} \overline{\left( \sum_{i=1}^n \lambda_i \omega_i \right)} - \int_{B_k} \left( \sum_{i=1}^n \lambda_i \omega_i \right) \int_{A_k} \overline{\left( \sum_{i=1}^n \lambda_i \omega_i \right)} \right]
\end{aligned}$$

Let  ${}^*\omega_i, {}^*\bar{\omega}_i$  be  $(\sum_{i=1}^n \lambda_i \omega_i), \overline{(\sum_{i=1}^n \lambda_i \omega_i)}$  by lemma 2.2.

$$\bar{\lambda}(-i\bar{\Omega}J^t\Omega)^t\bar{\lambda} = i \sum_{i=1}^n \left[ \int_{A_k} {}^*\omega_i \int_{B_k} {}^*\bar{\omega}_i \int_{B_k} {}^*\omega_i \int_{A_k} {}^*\bar{\omega}_i \right] > 0$$

Likewise theorem 2.1, if we choose a suitable base of holomorphic 1-forms, such that the period matrix  $\Omega = (\Pi, \Pi')$  is  $(I_n, T)$ . Thus we have

$$-i \begin{pmatrix} I_n & \bar{T} \end{pmatrix} \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} \begin{pmatrix} I_n \\ {}^tT \end{pmatrix} = i(\bar{T} - T) = 2ImT > 0$$

which proves lemma 2.2

q.e.d.



### 3 Period Matrix of $y^2 = x^{2n+1} - 1$

#### 3.1 The Way of Deciding Period Matrix of $y^2 = x^{2n+1} - 1$

##### 3.1.1 STEP 1: A base of holomorphic 1-forms for $y^p = x^q - 1$

**Lemma 3.1** Let  $N$  be the dimension of vector space of holomorphic 1-form  $\frac{x^b}{y^a}dx$  for  $y^p = x^q - 1$   $(p, q) = 1$ .

$$N = \frac{(p-1)(q-1)}{2}$$

Proof. (a)  $(x, y) = (\alpha_i, 0)$   $(i = 1, \dots, q)$

$\alpha_i$  is  $q$ -th root of 1. By  $py^{p-1}dy = qx^{q-1}dx$ , Order of  $dx$ 's zero point in  $y = 0$  is  $p-1$

$$\frac{x^b}{y^a}dx \text{ is holomorphic in (a)} \iff 1 \leq a \leq p-1$$

(b)  $(x, y) = (\infty, \infty)$

As we can view  $y^p = x^q - 1$  as  $y^p = x^q$  in (b), we can put  $x, y$  on  $t^{-p}, t^{-q}$  By  $dx = -pt^{-p-1}dt$ ,

$$\frac{x^b}{y^a}dx = \frac{t^{-bp}}{t^{-aq}}(-pt^{-p-1})dt = -pt^{aq-bp-(p+1)}dt$$

$$\frac{x^b}{y^a}dx \text{ is holomorphic in (b)} \iff aq - bp - (p+1) \geq 0$$

$$\iff 0 \leq b \leq \left[ \frac{aq - (p+1)}{p} \right]$$

From (a),(b)

$$\begin{aligned} N &= \sum_{a=1}^{p-1} \left[ \frac{aq - (p+1)}{p} \right] + 1 \\ &= \frac{1}{2} \sum_{a=1}^{p-1} \left[ \frac{aq - (p+1)}{p} \right] + 1 + \frac{1}{2} \sum_{a=1}^{p-1} \left[ \frac{(p-a)q - (p+1)}{p} \right] + 1 \\ &= \frac{1}{2} \sum_{a=1}^{p-1} (m-1) + 1 + \frac{1}{2} \sum_{a=1}^{p-1} (q-m-2) + 1 \quad aq = mp + r \quad (0 \leq r \leq p-1) \\ &= \frac{1}{2} \sum_{a=1}^{p-1} (q-1) = \frac{1}{2}(p-1)(q-1) \end{aligned}$$

which proves the lemma.

q.e.d.

### 3.1.2 STEP2 : Riemann Surface of $y^2 = x^{2n+1} - 1$

At first, let think about Riemann Surface of  $y^2 = x$  i.e.  $y = \sqrt{x}$ . We put the value of  $y$  for  $x_1 = re^{i\theta}$ ,  $x_2 = re^{i(\theta+2\pi)}$  on  $y_1$ ,  $y_2$ . As we can get  $y_2 = -y_1$  by easy calculation,  $y = \sqrt{x}$  is one-to-two mapping from  $x$ -surface to  $y$ -surface i.e. 2-valued-function which is showed by the below figure.

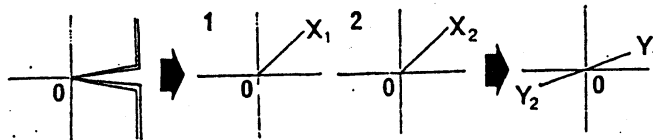


FIG 1

But  $y = \sqrt{x}$  is not 2-valued-function in  $x = 0$  which satisfy  $y = \sqrt{x} = 0$ . And As we can get  $t = \sqrt{s}$  again by putting  $x, y$  on  $1/s, 1/t$ .  $y = \sqrt{x}$  is not 2-valued-function in  $x = \infty$  too. Thus  $x = 0$  and  $x = \infty$  are branch points of  $R_0$ .  $R_0$  is the thing which joined up-side of  $x_1$ -surface to under-side of  $x_2$ -surface and under-side of  $x_1$ -surface to up-side of  $x_2$ -surface formally by cutting two surface  $x_1, x_2$  along segment connecting two branch points  $x = 0, \infty$ .

Next let think about Riemann Surface  $R_n$  of  $y^2 = x^{2n+1} - 1$  like  $y = \sqrt{x}$ .  $R_n$  has two sheets  $y_1 = \sqrt{x^{2n+1} - 1}$  and  $y_2 = -y_1 = -\sqrt{x^{2n+1} - 1}$ . As we can get  $t^2 = s^{2n+1}(1 - s^{2n+1})^{-1}$  by putting  $x, y$  on  $1/s, 1/t$  in  $y^2 = x^{2n+1} - 1$ , branch points of  $R_n$  are  $(2n+1)^{th}$ -root of 1  $x = p_i$  ( $i = 1, \dots, 2n+1$ ) and  $\infty$ . To see the model of  $R_n$ , we cut two sheets following so that the sheet can change by rounding each branch points one time.

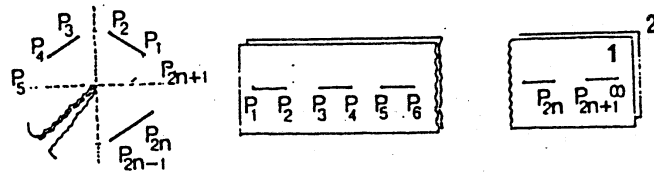


FIG 2

let join  $x_1$  - surface to  $x_2$  - surface like  $y = \sqrt{x}$ .

This is Riemann surface  $R_n$  of  $y^2 = x^{2n+1} - 1$ . Its genus  $g$  is the dimension of the vector space of holomorphic 1-forms by Riemann-Roch theorem. Therefore, by lemma 3.1.1  $g = 2\{(2n+1) - 1\}/2 = n$ .

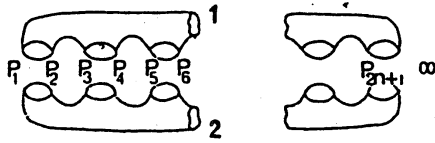


FIG 3

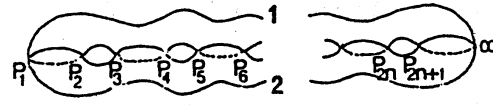
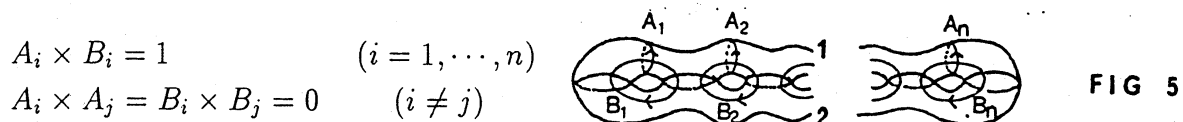


FIG 4

### 3.1.3 STEP 3: Five Rules of Deciding Period Matrix

We decide a base of homology for  $R_n$  which is made in Step 2. The following figure shows it.



Here, let's put the cross points of  $A_i, B_i$  on  $T_i$  ( $i = 1, \dots, n$ ) and we get together  $T_i$  to branch point  $\infty$  from same direction.

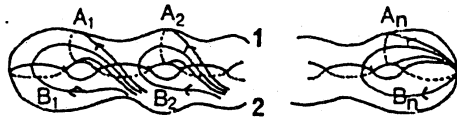


FIG 6

Next, we cut and open  $R_n$  along the base of homology  $A_i, B_i$ . It becomes a sheet of simple connected domain namely,  $4n$ -polynomial which has  $4g$ -side  $A_i^+, A_i^-, B_i^+, B_i^-$  ( $i = 1, \dots, n$ ) provided that  $A_i^+, B_i^+$  shows right side of  $A_i, B_i$  and  $A_i^-, B_i^-$  shows left side  $A_i, B_i$ . At last, we have to write difference of two sheet of complex surface  $x_1, x_2$  and arrangement of branch points in  $4n$ -polynomial.

On the above mentions, we decide the periods of holomorphic 1-forms for  $A_i, B_i$  by

On the above mentions, we decide the periods of holomorphic 1-forms for  $A_i, B_i$  by

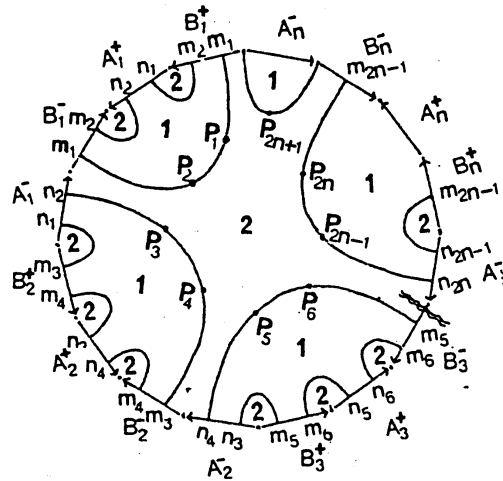


FIG 7

making some simple closed path which pass branch points. At this time, Cauchy's integral theorem plays the leading role. But the periods of holomorphic 1-forms for  $A_i, B_i$  must be decided uniquely so that they may satisfy Riemann's period relations i.e., Theorem 2.1 and Theorem 2.2. Therefore, to realize this object, we state the following five rules.

*In making a simple closed path which include the line  $A_i, B_i$*

- **Rule 1:** *A simple closed path must include right side  $A_i^-, B_i^-$ .*
- **Rule 2:** *A simple closed path must include even branch point.*
- **Rule 3:** *The sign of holomorphic 1-form in the path which get out from starting points of  $A_i^-, B_i^-$  must be same sign in the path which get into end points of  $A_i, B_i$ .*
- **Rule 4:** *The sign of holomorphic 1-form in the path which get out from a branch point must be different from sign in the path which get into same branch point.*
- **Rule 5:** *The sign of holomorphic 1-form in the path which connect two branch points must be unchangeable.*

**Remark 1:** As the side of  $4n$ -polynomial  $-B_i^-$  are enclosed by  $A_i^+$  and  $A_i^-$ . We can look on the path of  $B_i^-$  as the path which connect a point  $m_i$  on  $A_i^+$  and  $B_i^+$  by Cauchy's integral theorem. Thus we decide the periods of  $B_i$  for this new path by using above five rules.

**Remark 2:** As the side of  $4n$ -polynomial:  $A_i$  are enclosed by  $B_i^+$  and  $B_j^-$  ( $i \neq j$ ). We cannot choose a common point from the points on  $B_i^+$  and  $B_j^-$ . But at this case, we can decide value of period by making simple closed path which some pair of the side of  $4n$ -polynomial:  $A_i^+, A_i^-, B_i^+, B_i^-$ .

By using above five rules and two remarks, we decided periods of holomorphic 1-form. A base of holomorphic 1-forms for  $y^2 = x^{2n+1} - 1$  is following by lemma 3.1.1.

$$\omega_1 = \frac{1}{y}dx, \quad \omega_2 = \frac{x}{y}dx, \quad \omega_3 = \frac{x^2}{y}dx, \quad \dots, \quad \omega_n = \frac{x^{n-1}}{y}dx$$

Namely,

$$\omega_i = \frac{x^{i-1}}{y}dx = \frac{x^{i-1}}{\sqrt{x^{2n+1} - 1}}dx \quad (i = 1, \dots, n)$$

And let stand for  $\omega_i$  on  $x_1$ -surface,  $\omega_i$  on  $x_2$ -surface by  $\omega_{i1}, \omega_{i2}$ .

$$\omega_{i1} = \omega_i = \frac{x^{i-1}}{\sqrt{x^{2n+1} - 1}}dx, \quad \omega_{i2} = -\omega_i = \frac{-x^{i-1}}{\sqrt{x^{2n+1} - 1}}dx$$

## 3.2 Period matrix of $y^2 = x^{2n+1} - 1$

### 3.2.1 Calculation

#### 1. A - PERIOD MATRIX

(1) For  $j = 1, 2, \dots, n-1$

By Remark 2, we make a simple closed path which start from  $\infty$  in start point of  $A_j^-$ , pass the sides  $A_{j+1}^+ B_{j+1}^+ A_{j+1}^- B_{j+1}^-, A_{j+2}^+ B_{j+2}^+ A_{j+2}^- B_{j+2}^-, \dots, A_n^+ B_n^+ A_n^- B_n^-$ , arrive at  $\infty$  in start point of  $B_n^-$  and start from its  $\infty$ , pass the branch points  $P_1, P_2, \dots, P_{2j-1}, P_{2j}$ , come back  $\infty$  in end point of  $A_j^-$ . As the sum of the integrate values on sides of  $4n$  polynomial is 0 at this time, the integrate path of  $A_j$  is following.

$$\infty \xrightarrow{-} P_1 \xrightarrow{+} P_2 \xrightarrow{-} P_3 \xrightarrow{+} P_4 \cdots P_{2j-1} \xrightarrow{+} P_{2j} \xrightarrow{-} \infty$$

(2) For  $j = n$

We make the closed simple path which include  $A_n^-$  and pass the branch points  $P_1, P_2, P_3, P_4, \dots, P_{2n-1}, P_{2n}$ . Thus the integrate path of  $A_n$  is the following.

$$\infty \xrightarrow{-} P_1 \xrightarrow{+} P_2 \xrightarrow{-} P_3 \xrightarrow{+} P_4 \cdots P_{2n+1} \xrightarrow{+} P_{2n} \xrightarrow{-} \infty$$

We get period  $\pi_{ij}$  by the above integrate path

(1) For  $j = 1, 2, \dots, n-1$

$$\begin{aligned} \pi_{ij} &= \int_{A_j} \omega_i = \int_{A_{j+1}^+ B_{j+1}^+ A_{j+1}^- B_{j+1}^-} \omega_i + \int_{A_{j+2}^+ B_{j+2}^+ A_{j+2}^- B_{j+2}^-} \omega_i + \cdots + \int_{A_n B_n A_n B_n} \omega_i \\ &\quad + \int_{\infty}^{P_1} \omega_{i2} + \int_{P_1}^{P_2} \omega_{i1} + \int_{P_2}^{P_3} \omega_{i2} + \int_{P_3}^{P_4} \omega_{i1} + \cdots + \int_{P_{2j-1}}^{P_{2j}} \omega_{i1} + \int_{P_{2j}}^{P_{\infty}} \omega_{i2} \\ &= \int_{\infty}^{P_1} -\omega_i + \int_{P_1}^{P_2} \omega_i + \int_{P_2}^{P_3} -\omega_i + \int_{P_3}^{P_4} \omega_i + \cdots + \int_{P_{2j-1}}^{P_{2j}} \omega_i + \int_{P_{2j}}^{P_{\infty}} -\omega_i \\ &= 2(P_1^i - P_2^i + P_3^i - P_4^i + \cdots + P_{2j-1}^i - P_{2j}^i) \int_1^{\infty} \omega_i \end{aligned}$$

$$\pi_{ij} = 2 \sum_{l=1}^j (P_{2l-1}^i - P_{2l}^i) K_i$$

(2) For  $j = n$

$$\pi_{in} = \int_{B_n} \omega_i = \int_{\infty}^{P_1} -\omega_i + \int_{P_1}^{P_2} \omega_i + \int_{P_2}^{P_3} -\omega_i + \int_{P_3}^{P_4} \omega_i + \cdots + \int_{P_{2n+1}}^{P_{2n}} \omega_i + \int_{P_{2n}}^{P_{\infty}} -\omega_i$$

$$\pi_{in} = 2 \sum_{l=1}^n (P_{2l-1}^i - P_{2l}^i) K_i$$

## 2. B - PERIOD MATRIX

(1) For  $j = 1, 2, \dots, n-1$

By remark 1, we can view the path of  $B_j$  as the new path which connect the point  $m_{2j}^+$  on  $A_j^+$  and the point  $m_{2j}^-$  on  $A_j^-$ . Therefore, we make a simple closed paths which enclosed the new path and pass the branch points  $P_{2j}, P_{2j+1}$ . At this time, the integrate path of  $B_j$  is following.

$$m_{2j}^+ \xrightarrow{+} P_{2j+1} \xrightarrow{-} P_{2j} \xrightarrow{+} m_{2j}^-$$

(2) For  $j = n$

We make a closed simple path which include  $B_n^-$  and pass the branch points  $P_1, P_2, P_3, P_4, \dots, P_{2n}, P_{2n+1}$ . Thus the integrate path of  $B_n$  is following.

$$\infty \xrightarrow{+} P_{2n+1} \xrightarrow{-} P_{2n} \xrightarrow{+} \infty$$

We get period  $\pi'_{ij}$  by the above integrate path

(1) For  $j = 1, 2, \dots, n-1$

$$\begin{aligned} \pi'_{ij} &= \int_{B_j} \omega_i = \int_{m_{2j}^+}^{P_{2j+1}} \omega_{i1} + \int_{P_{2j+1}}^{P_{2j}} \omega_{i2} + \int_{P_{2j}}^{P_{2j+1}} \omega_{i1} = \int_{m_{2j}}^{P_{2j+1}} \omega_i + \int_{P_{2j+1}}^{P_{2j}} -\omega_i + \int_{P_{2j}}^{P_{2j+1}} \omega_i \\ &= 2(P_{2j}^i - P_{2j+1}^i) \int_1^\infty \omega_i \\ \pi'_{ij} &= 2(P_{2j}^i - P_{2j+1}^i) K_i \end{aligned}$$

(2) For  $j = n$

$$\begin{aligned} \pi'_{in} &= \int_{B_n} \omega_i = \int_\infty^{P_{2n+1}} \omega_{i1} + \int_{P_{2n+1}}^{P_{2n}} \omega_{i2} + \int_{P_{2n}}^\infty -\omega_{i1} \\ \pi'_{in} &= 2(P_{2n}^i - P_{2n+1}^i) K_i \end{aligned}$$

where,

$$\begin{aligned} P_i &= \exp\left(\frac{2\pi i}{2n+1}\right) \\ K_i &= \int_1^\infty \omega_i = \int_1^\infty \frac{1}{\sqrt{x^{2n+1}-1}} \end{aligned}$$

### 3.2.2 Result

#### 1.PERIOD MATRIX

A - period matrix  $\Pi$  :

By the above calculation,  $\pi_{ij} = 2 \sum_{l=1}^j (P_{2l-1}^i - P_{2l}^i) K_i \quad (i, j = 1, 2, \dots, n)$

$$\Pi = 2 \begin{pmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{pmatrix} \begin{pmatrix} P_1 - P_2 & P_1 - P_2 + P_3 - P_4 & \cdots & \sum_{l=1}^n (P_{2l-1} - P_{2l}) \\ P_1^2 - P_2^2 & P_1^2 - P_2^2 + P_3^2 - P_4^2 & \cdots & \sum_{l=1}^n (P_{2l-1}^2 - P_{2l}^2) \\ \vdots & \vdots & & \vdots \\ P_1^n - P_2^n & P_1^n - P_2^n + P_3^n - P_4^n & \cdots & \sum_{l=1}^n (P_{2l-1}^n - P_{2l}^n) \end{pmatrix}$$

B - period matrix  $\Pi'$  :

By the above calculation,  $\pi'_{ij} = 2(P_{2j}^i - P_{2j+1}^i)K_i \quad (i, j = 1, 2, \dots, n)$

$$\Pi' = 2 \begin{pmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{pmatrix} \begin{pmatrix} P_2 - P_3 & P_4 - P_5 & \cdots & P_{2n} - P_{2n+1} \\ P_2^2 - P_3^2 & P_4^2 - P_5^2 & \cdots & P_{2n}^2 - P_{2n+1}^2 \\ \vdots & \vdots & & \vdots \\ P_2^n - P_3^n & P_4^n - P_5^n & \cdots & P_{2n}^n - P_{2n+1}^n \end{pmatrix}$$

## 2. DETERMINANT

$$\det \Pi = 2K_1 K_2 \cdots K_n (P - P^2)(P^2 - P^4) \cdots (P^n - P^{2n}) H = 2(-1)^n K P^{\frac{n(n+1)(2n+1)}{6}} C$$

$$\det \Pi' = 2K_1 K_2 \cdots K_n (P^2 - P^3)(P^4 - P^6) \cdots (P^{2n} - P^{3n}) H = 2(-1)^n K P^{\frac{n(n+1)(n+2)}{3}} C$$

Let put  $C_k$  on  $\prod_{l=1}^k (P^{2l} - 1)$ , then H is following the Vandermonde determinant.

$$H = \begin{vmatrix} 1 & P^2 & P^4 & \cdots & P^{2n-2} \\ 1 & P^4 & P^8 & \cdots & P^{4n-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & P^{2n} & P^{4n} & \cdots & P^{2n^2-2n} \end{vmatrix} = P^{\frac{n(n-1)(n+1)}{3}} C_{n-1} C_{n-2} C_{n-3} \cdots C_2 C_1$$

$$K = K_1 K_2 \cdots K_n \quad K_i = \int_1^\infty \omega_i \quad C = C_1 C_2 \cdots C_{n-1} C_n, \quad C_n = \prod_{l=1}^n (P^l - 1)$$

$$P^i = P_i \quad (i = 1, 2, \dots, n) \quad P = \exp\left(\frac{2\pi i}{2n+1}\right)$$



### 3.2.3 Confirmation for lemma

In this section we show that the value of periods made by five rules satisfy Riemann's period relation 1.2.

$$\begin{aligned}
\sum_{k=1}^n \left[ \int_{A_k} \omega_i \int_{B_k} \omega_j \right] &= \sum_{k=1}^n \left[ \left\{ 2 \sum_{l=1}^k (P_{2l-1}^i - P_{2l}^i) K_i \right\} \left\{ 2(P_{2k}^j - P_{2k+1}^j) K_i \right\} \right] \\
&= 4K_i K_j \sum_{k=1}^n \left[ (1 - P^j) P_{2k}^j \sum_{l=1}^k \frac{1 - P^i}{P^i} P_{2l}^i \right] \\
&= 4K_i K_j \frac{(1 - P^i)(1 - P^j)}{P^i} \sum_{k=1}^n \left[ P_{2k}^j \sum_{l=1}^k P_{2l}^i \right] \\
&= 4K_i K_j \frac{(1 - P^i)(1 - P^j)}{P^i} \sum_{k=1}^n \left[ P_{2k}^j P_2^j \frac{1 - P_{2k}^i}{1 - P_2^j} \right] \\
&= 4K_i K_j \frac{1 - P^j}{1 + P^i} P^i \left\{ \sum_{k=1}^n P_{2k}^j - \sum_{k=1}^n P_{2k}^{i+j} \right\} \\
&= 4K_i K_j \frac{1 - P^j}{1 + P^i} P^i \left\{ P_2^j \frac{1 - P_2^{ni}}{1 - P_2^i} - P_2^{i+j} \frac{1 - P_2^{n(i+j)}}{1 - P_2^{i+j}} \right\} \\
&= -4K_i K_j \frac{1 - P^j}{1 + P^i} P^i \left\{ \frac{P^j}{1 + P^j} - \frac{P^{i+j}}{1 + P^{i+j}} \right\} \\
&= -4K_i K_j \frac{P^{i+j}(1 - P^i)(1 - P^j)}{(1 + P^{i+j})(1 + P^i)(1 + P^j)} \\
&\quad \text{(symmetric expression for } P^i, P^j)
\end{aligned}$$

$$\sum_{k=1}^n \left[ \int_{A_k} \omega_i \int_{B_k} \omega_j \right] = \sum_{k=1}^n \left[ \int_{A_k} \omega_j \int_{B_k} \omega_i \right]$$

which satisfy Riemann's period relation 1.

$$i \sum_{k=1}^n \left[ \int_{A_k} \omega_i \int_{B_k} \bar{\omega}_i - \int_{B_k} \omega_i \int_{A_k} \bar{\omega}_i \right] > 0$$

$$i \sum_{k=1}^n \left[ \overline{\int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i} - \int_{B_k} \omega_i \int_{A_k} \bar{\omega}_i \right] > 0$$

$$Im \sum_{k=1}^n \left( \int_{A_i} \bar{\omega}_i \int_{B_i} \omega_i \right) > 0$$

On the other hand, we have only to prove last inequality for Riemann's period relation 2.

$$\begin{aligned}
\sum_{k=1}^n \left[ \int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i \right] &= \sum_{k=1}^n \left[ \left\{ 2 \sum_{l=1}^k (\bar{P}_{2l-1}^i - \bar{P}_{2l}^i) K_i \right\} \left\{ 2(P_{2k}^i - P_{2k+1}^i) K_i \right\} \right] \\
&= 4K_i^2 (1 - \bar{P}^i)(1 - P^i) \sum_{k=1}^n \left[ \sum_{l=1}^k \frac{1}{\bar{P}^i} (\bar{P}_2^i)^l P_{2k}^i \right] \\
&= 4K_i^2 \frac{(1 - \bar{P}^i)(1 - P^i)}{\bar{P}^i} \sum_{k=1}^n \left[ \bar{P}_2^i \frac{1 - \bar{P}_{2k}^i}{1 - \bar{P}_2^i} P_{2k}^i \right] \\
&= 4K_i^2 \frac{1 - P^i}{1 + \bar{P}^i} \bar{P}^i \sum_{k=1}^n \left[ (1 - \bar{P}_{2k}^i) P_{2k}^i \right] \\
&= 4K_i^2 \frac{1 - P^i}{1 + \bar{P}^i} \bar{P}^i \sum_{k=1}^n \left[ P_{2k}^i - 1 \right] \\
&= 4K_i^2 \frac{1 - P^i}{1 + \bar{P}^i} \bar{P}^i \left\{ P_2^i \frac{1 - P_{2n}^i}{1 - P_2^i} - n \right\} \\
&= 4K_i^2 \frac{1 - P^i}{1 + \bar{P}^i} \bar{P}^i \left\{ \frac{-P^i}{1 + P^i} - n \right\} \\
&= 4K_i^2 \frac{(1 - \bar{P}^i)}{(1 + \bar{P}^i)(1 + P^i)} \{ P^i + n(1 + P^i) \} \\
\sum_{k=1}^n \left[ \int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i \right] &= \frac{4K_i^2 (1 - \bar{P}^i) \{ n + (n+1)P^i \}}{|1 + P^i|^2}
\end{aligned}$$

By putting

$$P^i = \exp\left(\frac{2\pi i}{2n+1}\right)$$

$$Im \sum_{k=1}^n \left[ \int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i \right] = \frac{4K_i^2}{|1 + P^i|^2} (2n+1) \sin\left(\frac{2\pi i}{2n+1}\right) > 0$$

which means that the period matrix for the homology in five rules satisfy Riemann's period relation. Thus we have the following theorem from the above result.

**THEOREM 3.1** *Let  $X$  be the Riemann surface defined by  $y^2 = x^{2n+1} - 1$ , then the period matrix of  $X$  is given by  $(\Pi, \Pi')$  of §3.2.2.*

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